A. Maths. I (B)

Unit (1)

Chapter.

Definite Integrals.

(निश्चित समाकलन)

(1) $\int_0^1 \frac{\log(1+x)}{1+x^2} dx = \frac{\pi}{8} \cdot \log 2$

Solution. $\int_0^1 \frac{\log(1+x)}{1+x^2} dx$

$$\left.\begin{aligned} x &= \tan\theta \\ dx &= \sec^2\theta \cdot d\theta \end{aligned}\right] \begin{matrix} 0 \\ \pi/4. \end{matrix}$$

$$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta} \cdot \sec^2\theta \cdot d\theta$$

$$= \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta} \cdot \sec^2\theta \cdot d\theta$$

$$I = \int_0^{\pi/4} \log(1+\tan\theta) \cdot d\theta$$

$$= \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right] d\theta$$

$$= \int_0^{\pi/4} \log\left[1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\cdot\tan\theta}\right] d\theta$$

$$= \int_0^{\pi/4} \log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right] d\theta$$

$$= \int_0^{\pi/4} \log\left[\frac{1+\tan\theta+1-\tan\theta}{1+\tan\theta}\right] d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\pi/4} \log 2 \cdot d\theta - \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$$

$$2I = \int_0^{\pi/4} \log 2 \cdot d\theta$$

$$= \log 2 \cdot \int_0^{\pi/4} 1 \cdot d\theta$$

$$= \log 2 \cdot [\theta]_0^{\pi/4}$$

$$= \frac{\pi}{4} \cdot \log 2$$

$$I = \frac{\pi}{8} \cdot \log 2$$

② $\int_0^{\pi} \frac{x \cdot dx}{a^2\cos^2 x + b^2 \sin^2 x} = \frac{\pi^2}{2ab}$

Solution. $I = \int_0^{\pi} \frac{x \cdot dx}{a^2\cos^2 x + b^2\sin^2 x}$

$$= \int_0^{\pi} \frac{(\pi - x)\, dx}{a^2\cos^2(\pi - x) + b^2\sin^2(\pi - x)}$$

$$= \int_0^{\pi} \frac{(\pi - x)\, dx}{a^2\cos^2 x + b^2\sin^2 x}$$

$$= \int_0^{\pi} \frac{\pi \cdot dx}{a^2\cos^2 x + b^2\sin^2 x} - \int_0^{\pi} \frac{x \cdot dx}{a^2\cos^2 x + b^2\sin^2 x}$$

$$2I = \pi \int_0^{\pi} \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$$

$$= \pi \int_0^{\pi} \frac{\sec^2 x \cdot dx}{a^2 + b^2 \cdot \tan^2 x}$$

$$= 2\pi \int_0^{\pi/2} \frac{\sec^2 x \cdot dx}{a^2 + b^2 \tan^2 x}$$

$$\left.\begin{aligned} b \cdot \tan x &= t \\ b \cdot \sec^2 x \cdot dx &= dt \end{aligned}\right] \begin{matrix} 0 \\ \infty \end{matrix}$$

$$= \frac{2\pi}{b} \int_0^{\infty} \frac{dt}{a^2 + t^2}$$

$$= \frac{2\pi}{b} \left[\frac{1}{a} \tan^{-1}\frac{t}{a}\right]_0^{\infty}$$

$$= \frac{2\pi}{ab} \left(\tan^{-1}\infty - \tan^{-1} 0\right)$$

$$= \frac{2\pi}{ab}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi^2}{ab}$$

$$I = \frac{\pi^2}{2ab}$$

Reduction Formula.

① $\int_0^{\pi/2} \sin^n x \cdot dx$ or, $\int_0^{\pi/2} \cos^n x \cdot dx$

$$= \frac{(n-1)(n-3)(n-5)\cdots}{n(n-2)(n-4)\cdots}, \quad \text{(n વિષમ હોય તો)}$$

$$= \frac{(n-1)(n-3)(n-5)\cdots}{n(n-2)(n-4)\cdots} \times \frac{\pi}{2}, \quad \text{(n સમ હોય તો)}$$

② $\int_0^{\pi/2} \sin^m x \cdot \cos^n x \cdot dx$

$$= \frac{(m-1)(m-3)(m-5)\cdots (n-1)(n-3)(n-5)\cdots}{(m+n)(m+n-2)(m+n-4)\cdots}$$

(m માંથી n માંથી કોઈ વિષમ હોય તો)

$$= \frac{(m-1)(m-3)(m-5)\cdots (n-1)(n-3)(n-5)\cdots}{(m+n)(m+n-2)(m+n-4)\cdots} \times \frac{\pi}{2}$$

(m તથા n કોઈ સમ હોય તો)

① $\int_0^{\pi/2} \sin^7 x \, dx$

$= \frac{(7-1)(7-3)(7-5)}{7(7-2)(7-4)(7-6)}$

$= \frac{6 \cdot 4 \cdot 2}{7 \cdot 5 \cdot 3 \cdot 1}$

$= \frac{16}{35}$

② $\int_0^{\pi/2} \sin^5 x \cdot \cos^6 x \cdot dx$

$= \frac{(5-1)(5-3) \cdot (6-1)(6-3)(6-5)}{11 \cdot (11-2)(11-4)(11-6)(11-8)(11-10)}$

$= \frac{4 \cdot 2 \cdot 5 \cdot 3 \cdot 1}{11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}$

$= \frac{8}{693}$

(3) $\int_0^{\pi} \theta \cdot \sin^6\theta \cdot \cos^4\theta \cdot d\theta$

$$I = \int_0^{\pi} \theta \cdot \sin^6\theta \cdot \cos^4\theta \cdot d\theta$$

$$= \int_0^{\pi} (\pi-\theta) \cdot \sin^6(\pi-\theta) \cdot \cos^4(\pi-\theta)\, d\theta$$

$$= \int_0^{\pi} (\pi-\theta) \cdot \sin^6\theta \cdot \cos^4\theta \cdot d\theta$$

$$= \pi \int_0^{\pi} \sin^6\theta \cdot \cos^4\theta \cdot d\theta - \int_0^{\pi} \theta \cdot \sin^6\theta \cdot \cos^4\theta \cdot d\theta$$

$$2I = \pi \int_0^{\pi} \sin^6\theta \cdot \cos^4\theta \cdot d\theta$$

$$= 2\pi \int_0^{\pi/2} \sin^6\theta \cdot \cos^4\theta \cdot d\theta$$

$$= 2\pi \cdot \frac{(6-1)(6-3)(6-5) \cdot (4-1)(4-3)}{10 \cdot (10-2)(10-4)(10-6)(10-8)} \cdot \pi/2$$

$$= 2\pi \cdot \frac{5 \cdot 3 \cdot 1 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \pi/2$$

$$= \frac{3\pi^2}{256}$$

$$I = \frac{3\pi^2}{512}$$